Transitional Flow in Concentric Annuli

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A general means is proposed for predicting transitional velocity patterns in concentric annuli and between parallel plates from behavior in tubes. As in laminar and turbulent flow, transformations of space coordinates yield identical profiles under specific conditions of comparison. Predictions are confirmed by published data and can be related to recent theoretical proposals.

The gross characteristics of laminar-turbulent transition in tubes have been described by Reynolds and others. Similar stages of behavior have been observed in other geometrical situations, yet the influence of boundary conditions has not been generalized. In that connection, the present work seeks a valid means of moving from one uniform duct to another when the boundaries are continuous, and the velocity pattern can be described in terms of a single, symmetrical length such as a radius.

Fully turbulent profiles in tubes, annuli, and parallel plates can be rendered coincident through a transformation of space coordinates (7 to 9). If, for instance, the tube is taken as the standard of comparison, that is the only shape for which a full array of primary information is required. Whether the data for tubes are based on theory or experiment makes no difference, provided they are accurate enough for the purpose at hand. If unrestricted by the type of motion or the time scale of eddies, the same approach can be made to transitional flow. A brief review of the underlying theory is, therefore, in order.

BASIC CONCEPTS

The pertinent conduit is a concentric annulus of inner radius r_1 and outer radius r_2 carrying a steady flow of Newtonian fluid having constant properties. The skin frictions at the inner and outer boundaries are $\tau_1 g_c$ and $\tau_2 g_c$, respectively, and at a particular radius r_m the local shearing stress τg_c is zero and the local, temporal-mean velocity u is at its maximum value u_m. Regardless of the type of flow, the distribution of shearing stress is linear in the hydraulic radius R_H written over the portion of the stream between radius r_m and any other arbitrary radius r. At boundary s (either inner surface 1 or outer surface 2), the radius is r_s and the skin friction is $\tau_s g_c$. Thus

$$\frac{\tau g_c}{\tau_s g_c} = \frac{\left(\dfrac{r^2 - r_m^2}{2r}\right)}{\left(\dfrac{r_s^2 - r_m^2}{2r_s}\right)} = \frac{R_H}{R_{H_s}} \tag{1}$$

For convenience, the relationship between stress and strain can be written in terms of the eddy viscosity ϵ as

$$\tau g_c = -(\mu + \epsilon)\frac{du}{dr} \tag{2}$$

Since ϵ is simply a parameter whose value is zero for laminar flow, the form of Equation (2) can be retained for all types of flow. The velocity-mean value of ϵ can be introduced through the definitive relationship

$$\epsilon^0 u = \int_0^u \epsilon \, du \tag{3}$$

By combining the three equations and by integrating from the boundary radius r_s to the arbitrary radius r or from r_s to the radius of maximum velocity r_m as the case may be, the following results are obtained:

$$\left(\frac{\Phi_m}{\Phi}\right)_s \frac{[1 + (\epsilon^0/\mu)]u}{\Phi_{ms} R_{H_s} \tau_s g_c/\mu} = \frac{1}{2} = \frac{[1 + (\epsilon_m^0/\mu)]u_m}{\Phi_{ms} R_{H_s} \tau_s g_c/\mu} \tag{4}$$

where

$$\Phi_s = \frac{2}{R^2_{H_s}} \int_r^{r_s} R_H \, dr \tag{5}$$

and

$$\Phi_{ms} = \frac{2}{R^2_{H_s}} \int_{r_m}^{r_s} R_H \, dr \tag{6}$$

The results of integrating the last two equations over various cross sections are summarized in Table 1. The position parameters Φ_s and Φ_{ms} are geometrical in nature but depend on Reynolds number over any range of flow in which r_m changes with Reynolds number. It is immediately apparent from Equation (4) that

$$u/u_m = (\Phi/\Phi_m)_s \left(1 + \frac{\epsilon_m^0}{\mu}\right) \Big/ \left(1 + \frac{\epsilon^0}{\mu}\right) \tag{4a}$$

so for laminar flow

$$u/u_m = (\Phi/\Phi_m)_s \tag{4b}$$

Reduced velocity profiles are, therefore, coincident on the transformed coordinates in laminar motion, and it remains to specify conditions under which they might also be expected to coincide in transitional and turbulent flow.

It is reasonable to define a Reynolds number and friction factor as follows:

$$N_{Re_s} = 2\,\Phi_{ms} R_{H_s} V\rho/\mu \tag{7}$$

$$f_s = 2\,\tau_s g_c/\rho V^2 \tag{8}$$

Equations (4), (7), and (8) then yield

$$[1 + (\epsilon_m^0/\mu)] = \eta_s/8 = (u/u_m)(\Phi_m/\Phi)_s[1 + (\epsilon^0/\mu)] \tag{9}$$

where

$$\eta_s = N_{Re_s} f_s (V/u_m)_a \tag{10}$$

Together, Equations (9) and (10) imply that profiles of u/u_m against $(\Phi/\Phi_m)_s$ are coincident in any annuli having equal values of $[N_{Re_s} f_s (V/u_m)_a]$ and that the absolute velocities are coincident in the same space when the maximum velocities are also equal. The criterion of comparison is independent of the flow regime and, therefore, should be equally applicable to laminar, transitional, and fully turbulent cases. Whether corresponding velocities are actually attained at equal values of η_s, however, must be established by experiment.

COMPARISON WITH SMOOTH TUBES

Since velocity data for tubes are more plentiful than for other cross sections, it is reasonable to make tubes the basis of comparison. For present purposes, only smooth conduits are considered.

By virtue of Equation (9), it is clear that the parameter η_s must have a constant numerical value, namely 8, when the flow is entirely laminar. When the eddy vis-

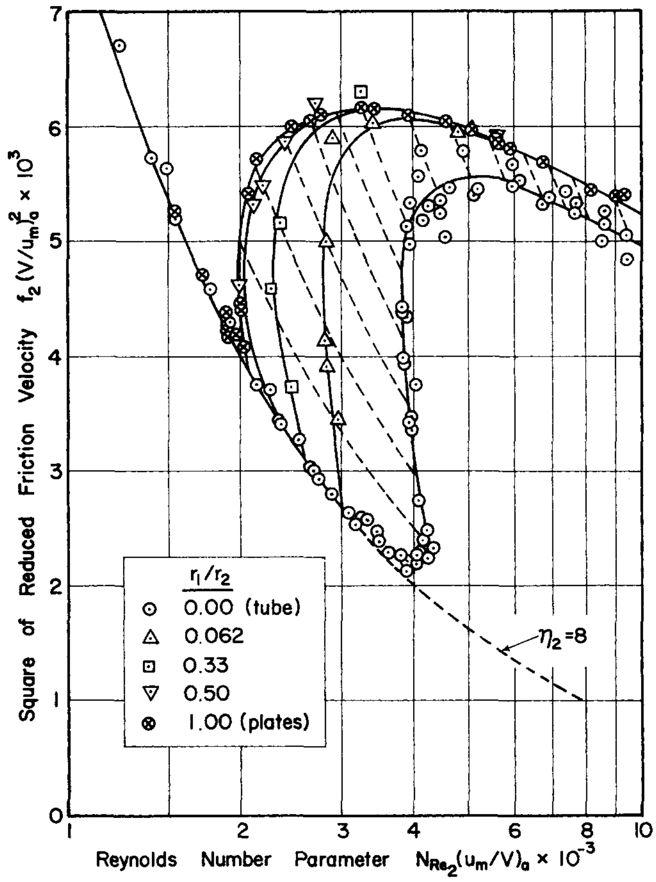

Fig. 1. Influence of Reynolds number on reduced friction velocity in the outer portion of annuli having various radius ratios. (Dashed tie-lines connect equal values of the criterion parameter η_2.)

cosity is other than zero at any point in the stream, however, the parameter must have a greater value, functionally connected with the Reynolds number. To date, experimental information is ordinarily needed to demonstrate the relationship.

It is also clear from Equation (9) that the portions of the annular stream on either side of the point of maximum velocity should be treated separately. There is no a priori reason to believe that the inner and outer portions can be represented by the same equivalent tube. It turns out, in fact, that the equivalent tubes must operate at different Reynolds numbers, even when their sizes are the same. In view of that, a certain tube t can be chosen to represent the portion of the annulus between the radius of maximum velocity r_m and the radius of the outer wall r_2. Another tube p can similarly be taken to represent the portion between r_m and the radius of the inner wall r_1. Then, the separate criteria of comparison can be written as

$$\eta_t = N_{Ret} f_t (V/u_m)_t = N_{Re2} f_2 (V/u_m)_a = \eta_2 \quad (11)$$

$$\eta_p = N_{Re_p} f_p (V/u_m)_p = N_{Re1} f_1 (V/u_m)_a = \eta_1 \quad (12)$$

The subscript a, of course, simply refers to the annulus as a whole, since the bulk average velocity V is computed over the entire cross section.

The friction factors f_1 and f_2 stand in the ratio $R_{H1}:R_{H2}$, so, upon applying Equation (7) to the separate boundaries, it can be seen that

$$\frac{\eta_p}{\eta_t} = \left(\frac{\Phi_{m1}}{\Phi_{m2}}\right)\left(\frac{R_{H1}}{R_{H2}}\right)^2 = \frac{\eta_1}{\eta_2} \quad (13)$$

Since η_p and η_t are functions of Reynolds number alone, the Reynolds number of tube p is immediately established by the last equation, once that of tube t has been obtained. When the radius of maximum velocity has its laminar-flow value, namely (6)

$$r_{mL} = [(r_2^2 - r_1^2)/2 \ln (r_2/r_1)]^{1/2} \quad (14)$$

it turns out that $\eta_p = \eta_t$, making $N_{Re_p} = N_{Ret}$. For any other value of r_m, however, the equivalent tubes operate at different Reynolds numbers.

TREATMENT OF TURBULENT FLOW

When the flow is fully turbulent, it proves most convenient to write Equation (11) in the form

$$\eta_t = [N_{Ret} (u_m/V)_t] \, [f_t (V/u_m)_t^2]$$
$$= [N_{Re_2} (u_m/V)_a] \, [f_2 (V/u_m)_a^2] = \eta_2$$

and to deal separately with the two kinds of bracketed terms. Thus, for equal skin frictions and equal maximum velocities in the tube and annulus, criteria of comparison can be written as

$$\sqrt{f} \, (V/u_m) = \text{constant} \quad (15)$$

$$N_{Re} \, (u_m/V) \text{ or } N_{Re} \sqrt{f} = \text{constant} \quad (16)$$

They assert that the relationship between $f (V/u_m)^2$ and $N_{Re} (u_m/V)$ for tubes is equally valid for annuli. Together, the separate criteria are more stringent than the single criterion represented by Equations (11) or (12). Provided the Reynolds number is high enough (say, greater than 30,000), experimental data show that use of the separate criteria stated in Equations (15) and (16) is accurate as well as convenient (*8*).

TREATMENT OF TRANSITIONAL FLOW

When the flow is transitional, the reduced friction velocity is not a unique function of Reynolds number from annulus to annulus as it is in full turbulence. Consequently, Equations (15) and (16) are not separately applicable, and the criterion of comparison reverts to the less restrictive statement represented by Equations (11) or (12). The proposition to be tested by experimental data is, therefore, the following.

When a tube and annulus are compared under conditions such that the dimensionless parameter η_s is numerically the same in both ducts, the reduced velocity profiles are coincident on (u/u_m) against $(\Phi/\Phi_m)_s$ coordinates regardless of the type of flow. Portions of the annulus inside and outside the radius of maximum velocity are handled separately, but the basic criterion of comparison is unchanged.

SOURCES OF DATA

In order to test the proposition, data were required in terms of friction factors and velocity profiles for tubes, parallel plates, and annuli of various radius ratios. The criterion of comparison was known to be theoretically valid in laminar flow and already verified at high Reynolds numbers, so attention was centered on data for the heretofore untested transitional range.

Friction factors, V/u_m values, and velocity profiles for smooth tubes were obtained from the data of Senecal (*11, 12*). Corresponding values for parallel plates were from the data of Whan (*14, 15*). Similar data on annuli having radius ratios of 0.062, 0.33, and 0.50 were taken from the results of Walker (*13*) and Croop (*1, 2*). Separate information on the change of position of maximum velocity with Reynolds number was obtained from measurements by Walker (*13*), Croop (*1*), and Sartory (*10*).

Table 1. Position Parameters for Tubes, Annuli, and Parallel Plates

Conduit	Φ_s	Φ_{ms}	$(\Phi/\Phi_m)_s$
Tube (radius r_0)	$2\left[1-\left(\frac{r}{r_0}\right)^2\right]$	2.0	$1-\left(\frac{r}{r_0}\right)^2$
Parallel plates (clearance 2b)	$1-\left(\frac{r}{b}\right)^2$	1.0	$1-\left(\frac{r}{b}\right)^2$
Annulus, inner portion (r_1 to r_m)	$\frac{1}{2R_{H1}^2}\left[r_1^2-r^2-2r_m^2\ln\left(\frac{r_1}{r}\right)\right]$	$\frac{1}{2R_{H1}^2}\left[r_1^2-r_m^2-2r_m^2\ln\left(\frac{r_1}{r_m}\right)\right]$	$\frac{r_1^2-r^2-2r_m^2\ln\left(\frac{r_1}{r}\right)}{r_1^2-r_m^2-2r_m^2\ln\left(\frac{r_1}{r_m}\right)}$
Annulus, outer portion (r_m to r_2)	$\frac{1}{2R_{H2}^2}\left[r_2^2-r^2-2r_m^2\ln\left(\frac{r_2}{r}\right)\right]$	$\frac{1}{2R_{H2}^2}\left[r_2^2-r_m^2-2r_m^2\ln\left(\frac{r_2}{r_m}\right)\right]$	$\frac{r_2^2-r^2-2r_m^2\ln\left(\frac{r_2}{r}\right)}{r_2^2-r_m^2-2r_m^2\ln\left(\frac{r_2}{r_m}\right)}$

In all cases, the data were used in their original, unsmoothed form. The results presented herein are, therefore, not biased by previous smoothing and reflect the precision of correlation to be expected when working directly with typical experimental determinations.

RESULTS AND DISCUSSION

Figure 1 compares various cross sections with respect to the influence of Reynolds number on the reduced friction velocity. Of course, the abscissa N_{Re2} (u_m/V) and the ordinate f_2 $(V/u_m)^2$ yield the parameter η_2 when multiplied by one another. It is, therefore, apparent that the separate criteria of Equations (15) and (16) are useful only at sufficiently low or sufficiently high Reynolds numbers, where the curves for different radius ratios are coincident or nearly so. In the intermediate range covering transition, tie lines are needed to connect equal values of the ordinate times the abscissa.

Figure 1 also illustrates typical tie lines connecting equal values of η_2 for tubes, parallel plates, and the outer portion of annuli. Were the coordinates Cartesian, the tie lines would, of course, be hyperbolic in shape. Lines of constant radius ratio are also shown, along with the data supporting their empirical position. It is clear that any annuli having radius ratios greater than about 0.1 resemble parallel plates more closely than tubes in the transitional range. It is noteworthy that the Reynolds number alone does not serve to define the equivalent tube uniquely in transition, so frictional data are a necessary supplement. That is, more than one value of the ordinate can be read along a line of constant radius ratio at a given value of the abscissa, so the ordinate must be established before the proper tie line can be identified.

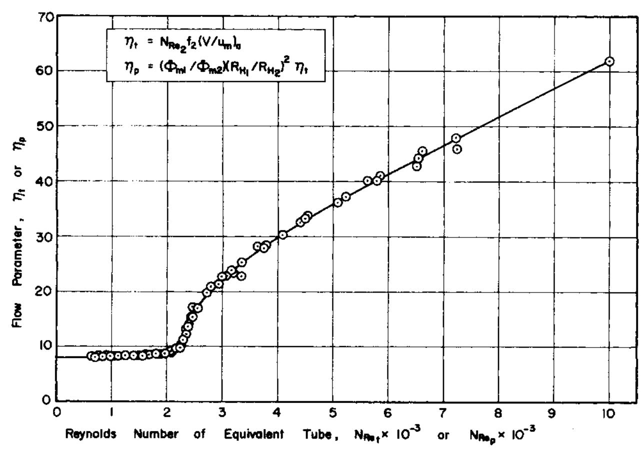

Fig. 2. Relationship between the criterion parameter for an annulus and the Reynolds number of the equivalent tube.

The information conveyed by Figure 1 can be translated into the simple form presented as Figure 2. When the abscissa is taken to be the parameter η_2, the ordinate is N_{Re_t}, the Reynolds number of the tube equivalent to the outer portion of the annulus. When the abscissa is read as η_1, the ordinate is N_{Re_p}, the Reynolds number of the tube equivalent to the inner portion of the annulus.

The procedure is straightforward. The parameter η_2 is computed from annular data, since by virtue of Equation (11)

$$\eta_2 = N_{Re2} f_2 \, (V/u_m)_a$$

The Reynolds number N_{Re_t} of the equivalent tube is then read directly from Figure 2. With η_2 available, η_1 can be calculated by means of Equation (13):

$$\eta_1 = \eta_2 \left(\frac{\Phi_{m1}}{\Phi_{m2}}\right)\left(\frac{R_{H1}}{R_{H2}}\right)^2$$

The Reynolds number N_{Re_p} of the corresponding equivalent tube can again be read from Figure 2. That completes the determination of Reynolds numbers, and it remains only to obtain the corresponding velocity profiles from tubular data.

When the velocity profiles for tubes are available as curves of u/u_m against y/r_0 at constant Reynolds number, it is most convenient to replot them as u/u_m against $1 - (r/r_0)^2$ by using the identity

$$1-\left(\frac{r}{r_0}\right)^2 = \frac{y}{r_0}\left(2-\frac{y}{r_0}\right)$$

The profiles are now graphs of u/u_m against Φ/Φ_m for tubes and are coincident with the same curves for that portion of the annulus under consideration. Thus, at a particular position r in the annular space, the value of Φ/Φ_m can be obtained from the equations of Table 1 provided r_m is known, and the corresponding u/u_m can then be read directly from the tubular profile at the Reynolds number of the equivalent tube. The same procedure is followed on each side of the maximum point, so the whole annular velocity profile can be predicted.

Figures 3 and 4 illustrate the extent to which predicted profiles are supported by experimental data. Figure 3 is included to show that the data do not reproduce the theoretical profile perfectly in laminar flow. Points nearest the wall are least reliable and tend to be low. Figure 4 indicates that transitional and laminar data are consist-

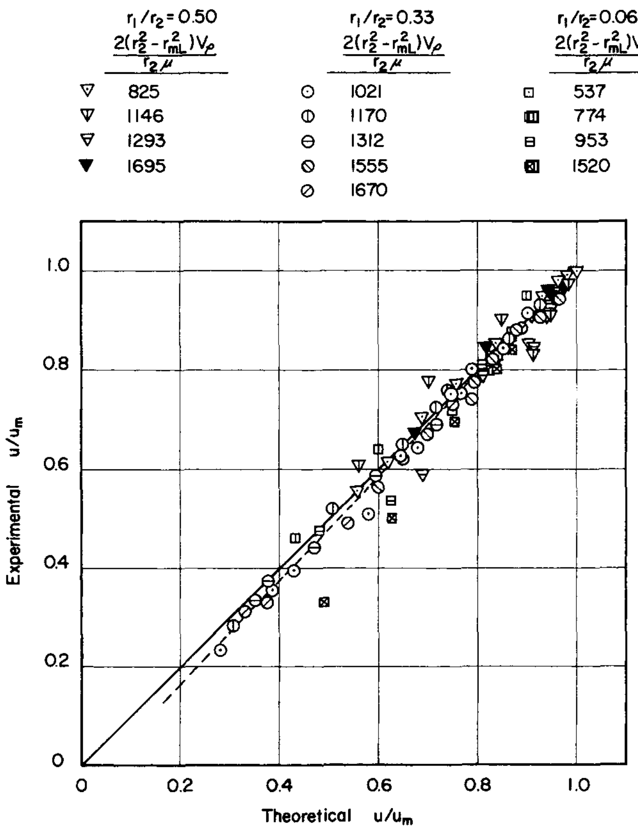

Fig. 3. Comparison of experimental data with theoretical velocity profiles; fully laminar range. (Dashed line showing trend of data is reproduced in Figure 4.)

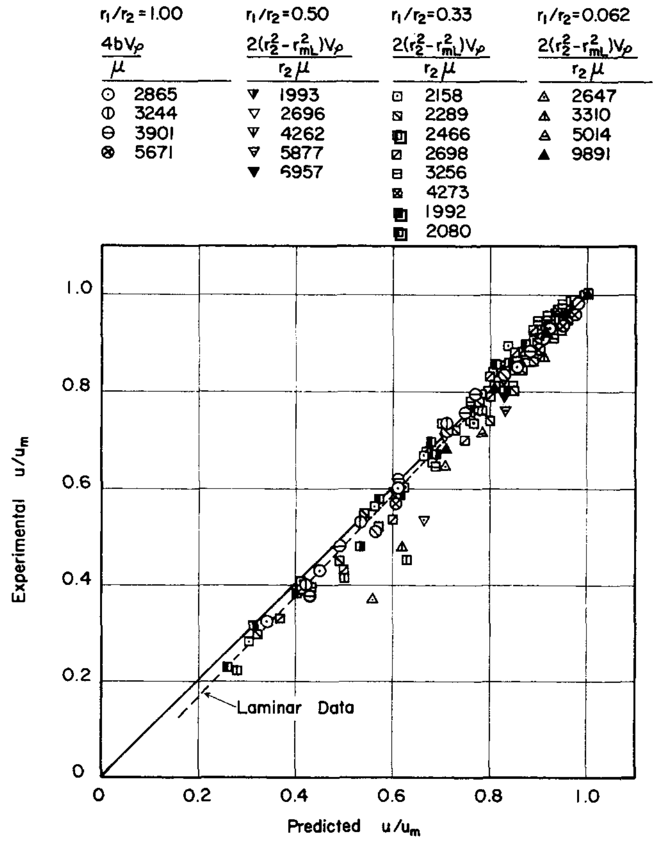

Fig. 4. Comparison of experimental velocity data with predictions based on the present work; transitional range. (Dashed line representing laminar deviation is also followed in transitional flow.)

ent in that respect and that most of the apparent deviation from the diagonal is due to experimental error rather than shortcomings of the prediction. Although conclusions must be tempered by the quality of data, it seems fair to say that Figure 4 fully confirms predicted profiles within anticipated precision.

It is important to recognize that the present criterion says nothing about the critical Reynolds number marking the onset of disturbance eddies. That is a separate matter. The lines of constant radius ratio in Figure 1 have been terminated at the positions along the laminar curve ($\eta_2 = 8$) corresponding to the theory proposed by Hanks (*4*). Although there are no experimental points in the immediate vicinity of the critical Reynolds numbers, the data can be extrapolated to Hanks' predicted values without difficulty and are apparently in good agreement with them.

Hanks (*5*) has very recently proposed a theoretical means of treating one of the constants in a mixing length expression set forth by Van Driest (*16*) and modified by Gill and Scher (*3*). For a tube, the mixing length L is taken to be

$$L/r_0 = k\,(y/r_0)\,[1 - \exp\,(\phi y/r_0)] \tag{17}$$

where, according to Hanks

$$\begin{aligned}\phi &= [N_{Re}\sqrt{f} - (N_{Re}\sqrt{f})_c]/2B\sqrt{2}\\ &= [N_{Re}\sqrt{f} - 183.3]/2B\sqrt{2}\end{aligned} \tag{18}$$

B is a penetration parameter with a value of about 22 or 23, and k is Prandtl's universal constant, often taken to be 0.36.

Since annuli and tubes can be compared at constant η_s regardless of the form of the tubular data, Equations (17) and (18) can be generalized on that basis by taking

$$y/r_0 = 1 - \sqrt{1 - (\Phi/\Phi_m)_s} \tag{19}$$

and

$$N_{Re}\sqrt{f} = \eta_s/(V/u_m)_i\sqrt{f_i} \tag{20}$$

where $i = t$ when $s = 2$ and $i = p$ when $s = 1$. It is easy to show that Equations (18) and (20) are in exact agreement with Hank's formulation of the damping factor ϕ for the limiting case of flow between parallel plates. Again, however, it must be emphasized that since the parameter η_s includes the product of a Reynolds number and a friction factor, it can offer no information about the critical Reynolds number. Suppose it is desired to construct one of the curves of Figure 1 at a specified radius ratio from the mixing length expression without further velocity data. The critical Reynolds number must be fixed to provide a starting point for the calculation. Therefore, a critical criterion such as the one proposed by Hanks is needed initially in order to separate the Reynolds number from the friction velocity. Once that has been done, Equations (17) to (20) provide the basis for calculating the rest of the desired relationship. In the absence of empirical data, iterative computation is needed to evolve ratios of average to maximum velocity. For the time being, the radius r_m must be established experimentally over the transitional range.

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NOTATION

b = half clearance between parallel plates, ft.

B = penetration parameter in Equation (17), dimensionless
f = Fanning friction factor defined by Equation (8), dimensionless
g_c = conversion factor = 32.2 (lb. mass) (ft.)/(lb. force) (sec.^{2})
k = universal constant in Equation (17), dimensionless
L = Prandtl mixing length, ft.
N_{Re} = Reynolds number defined by Equation (7), dimensionless
r = radius to an arbitrary point in the fluid, ft.
r_m = radius of maximum velocity, ft.
r_0 = radius of tube, ft.
R_H = hydraulic radius for the portion of the stream between r_m and r, ft.
R_{H_s} = hydraulic radius for the portion of the stream between r_m and the boundary s, ft.
u = local fluid velocity at radius r, ft./sec.
u_m = maximum local velocity of fluid, ft./sec.
V = bulk average linear velocity of fluid, ft./sec.
y = normal distance from wall of conduit, ft.

Greek Letters

ϵ = local eddy viscosity at radius r, lb. mass/(sec.) (ft.)
ϵ^0 = velocity-mean eddy viscosity between the boundary s and radius r, lb. mass/(sec.) (ft.)
$\epsilon_m{}^0$ = velocity-mean eddy viscosity between the boundary s and the radius of maximum velocity, lb. mass/(sec.) (ft.)
η_s = criterion parameter defined by Equation (10), dimensionless
μ = fluid viscosity, lb. mass/(sec.) (ft.)
ρ = fluid density, cu. ft./lb. mass
τ = local shearing stress, lb. force/sq. ft.
τ_s = skin friction at boundary s, lb. force/sq. ft.
ϕ = damping factor in Equation (17), dimensionless
Φ = distance parameter defined in Equation (5), dimensionless
Φ_m = distance parameter defined in Equation (6), dimensionless

Subscripts

1 = inner boundary of annulus or portion of stream between r_m and r_1
2 = outer boundary of annulus or portion of stream between r_m and r_2
i = equivalent tube t or p, whichever is pertinent
L = laminar flow
m = maximum velocity or its position
p = tube equivalent to inner portion of annulus between r_m and r_1
s = boundary s or portion of stream between r_m and r_s
t = tube equivalent to outer portion of annulus between r_m and r_2
c = critical value at onset of disturbance eddies

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The Structuring of Process Optimization

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The efficiency of process optimization by mathematical programming can be increased by tearing, that is, rearranging the design equations so as to reduce the number of equality constraints. The structure of a system of equations may be depicted as an undirected bipartite graph; algorithm I-T utilizes this graph to determine an order of solution for the equations which requires no tears. If this is impossible, then algorithm II-T uses indexing in conjunction with algorithm I-T to produce an order which minimizes the number of torn equations. This procedure is extended to the problem of minimum recycle parameters, and the two-way interaction between tearing and algebraic simplification is illustrated.

THE STRUCTURING OF PROCESS OPTIMIZATION

The problem of optimizing a process design may be stated in general algebraic form as:
Maximize the objective function

$$z(x_1, x_2, \ldots, x_n) = z(\mathbf{x}) \quad (1)$$

subject to the equality constraints

$$f_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, m$$

or

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \quad (2)$$

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